CONCEPTS OF INTERVAL ESTIMATION AND QUALITY CONTROL CHARTS VIA COMPUTER SIMULATED SAMPLING

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ABSTRACT

The interconnection between confidence interval estimation and statistical decision making with control charts is discussed. The concepts of confidence in interval estimation and statistical process control are presented with the aid of plots of the results of simulated sampling. Plots of confidence intervals with known standard deviations and with estimated standard deviations and a control chart are presented. Data that were used by Gosset (1908) with the introduction of the t-distributions are used for an example with historical import.

INTRODUCTION

Quality control is an important area of statistical practice. To compete effectively in the global marketplace, quality control is used to manage and constantly improve processes for the production of goods and services. In the present corporate environment, managers must apply existing quality technologies and become knowledgeable about new developments in order to improve quality and increase productivity. The production of goods and services usually involves continuing processes. To manage and constantly improve a process, managers may employ methods of statistical quality control to examine characteristics of the process over time in order to reduce process variability and enhance process performance. This approach is based on the collection and analysis of process data.

Statistical process control is a methodology using graphical displays known as control charts for assistance in monitoring quality of conformance and eliminating special causes of variability in a process (Evans & Lindsay, 1989). Process characteristics, such as the mean or variation of a process, are typically monitored with control charts. A control chart is a plot of descriptive measures obtained for a process against time; furthermore, a centerline for the process characteristic and lower and upper control limits are also included on the chart. On the $\bar{X}$ control chart the control limits are set so that there is only a very small probability that a sample mean will fall outside of the control limits when the process is in fact operating at the specified average on the control chart; thus, control limits are expected to encompass essentially all of the values of a descriptive measure that result from subgroups of data taken periodically from a stable process. Control limits are determined by using measures of process variation and are based on the concepts surrounding hypothesis testing and interval estimation. Thus, the concept of interval estimation is a method for statistical inference that is important for decision making in general, but it is inherently linked to the nature of quality control decisions as well. Interval estimation allows managers to account for uncertainty and variation during the analysis of processes. Consequently, it is important for managers to understand the concept of confidence in the context of interval estimation and the connections to quality control.

Confidence interval estimation can be founded on formal probability theory and confidence intervals can be derived mathematically (Hogg & Craig, 1978). Nevertheless, understanding of concepts of confidence can be enhanced by using simulation to show actual results of processes of statistical inference. Furthermore, the results of simulations can be shown concretely by using plots of the outcomes.

The purpose of this paper is to demonstrate the concept of confidence in interval estimation by using simulations and thereby help students gain a broader understanding of the role and value of statistical decision making and to link interval estimation with statistical quality control and control charts. This is accomplished by presenting the results of the simulations with the aid of computer-produced plots. Simulations of confidence interval estimation of a mean with known and with unknown standard deviations are presented in two examples. The software used for the simulations and for producing the plots is available upon request.

Example 1

The first example deals with the problem of estimating the mean of a normal population when the variance sigma^2 is known. Assuming that sigma is known acts to facilitate understanding of the reasoning and clarifies the principles underlying estimation; it also introduces the more complicated and more realistic case, treated in the next example, where sigma is unknown.

Gosset, writing under the pseudonym Student (Student, 1908), estimated the effectiveness a drug in terms of the mean amount of increase in sleep that patients might expect. Ten patients were given the sleep-enhancing drug. In almost every case the patient slept longer under the effect of the drug than usual; Table I shows the amount of the increase in sleep (in hours) in each case. The amount of increase in sleep varies, so we want to estimate the increase in such a way that the reliability of our conclusions concerning the effects of the drug may be evaluated by means of probability or confidence. Thus we want to find a confidence interval estimate for the mean increase in sleep in hours that we might expect from the drug (Watson et al., 1986).

A confidence interval is an interval, bounded on the left by L and on the right by R, that is used to estimate an unknown population parameter. The interval is constructed in such a way that the reliability of the estimate may be evaluated objectively by means of a confidence statement.

The terms L and R are used for confidence limits. The 100(1 - alpha)% confidence interval for the population mean mu when the population is normally distributed and the variance sigma^2 is known is the interval bounded by the confidence limits as follows:

Using the sleep enhancing drug data, assume we know from past experience that the increase in sleep is normally

$$L = \bar{X} - Z_{alpha/2}\ sigma/n^{1/2} \qquad (1)$$

$$R = \bar{X} + Z_{alpha/2}\ sigma/n^{1/2} \qquad (2)$$

distributed with some mean mu and the population variance is $sigma^2 = 1.66$. The estimator X then has the following variance:

$$sigma^2\bar{X} = \frac{sigma^2}{n} = \frac{sigma^2}{10} = \frac{1.66}{10} = .166 \qquad (3)$$

and it has the following standard deviation:

$$sigma\bar{X} = \frac{sigma}{(n)^{1/2}} = \frac{(1.66)^{1/2}}{(10)^{1/2}} \qquad (4)$$

We want to use a confidence level of 95% and the data of Table 1 to find an interval estimate for the mean amount of increase in sleep mu. In other words, we want to find the 95% confidence interval estimate for the mean increase in sleep.

Since the confidence level is 100(1- alpha) = 95%, we have

$$1 - alpha = .95 \quad alpha = .05 \qquad (5)$$

$$alpha/2 = .025 \quad Z_{.025} = 1.96 \qquad (6)$$

Now the data of Table 1 for the ten patients give X = 1.58 hours, and we know that sigmaX = sigma/(n)”2 = (1.66)1/27(10)1/2. Thus we have confidence limits of:

$$L = \bar{X} - Z_{alpha/2}\frac{sigma}{(n)^{1/2}} = \bar{X} - 1.96\frac{(1.66)^{1/2}}{(10)^{1/2}}$$

$$= 1.58 - .8 = .78 \qquad (7)$$

and

$$R = \bar{X} + Z_{alpha/2}\frac{sigma}{(n)^{1/2}} = \bar{X} + 1.96\frac{(1.66)^{1/2}}{(10)^{1/2}}$$

$$= 1.58 + .8 = 2.38 \qquad (8)$$

Thus we can feel 95% confident that the population mean lies between .78 and 2.38.

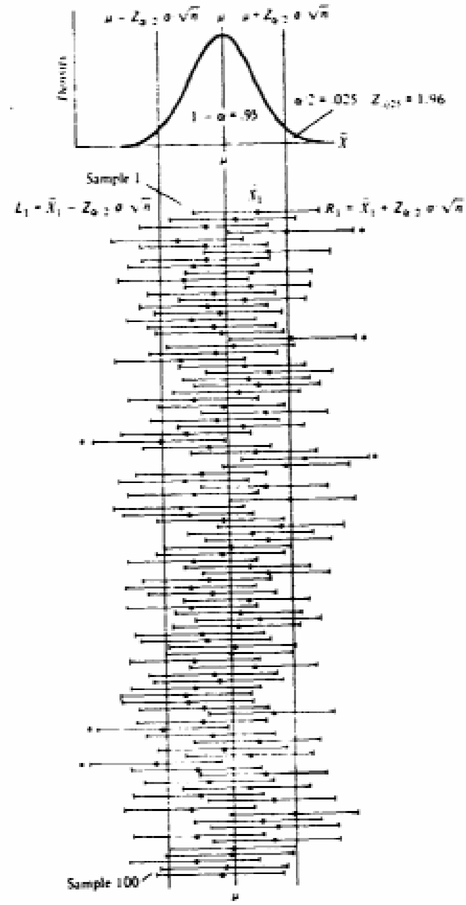
The confidence we have in the limits .78 and 2.38 in Example I derives from our confidence in the statistical procedure that gave rise to them. The procedure gives random variables L and R that have a 95% chance of enclosing the true but unknown mean mu; whether their specific values .78 and 2.38 enclose mu we have no way of knowing.

The meaning of our being 95% confident is shown in the results of a simulation. We took 100 different random samples of size n = 10 each from a normal population with a mean mu and a standard deviation sigma = (1.66)1/2 One hundred sample means, X’s, and corresponding confidence intervals were then computed, and the results are presented in Figure 1. The figure shows that the mean mu was contained in 94 of the 100 intervals. This result conforms to our expectation that about 95 of 100 intervals should encompass the mean mu (Watson et al., 1986).

Source: Watson, C., Billingsley, P., Croft, J., Huntsberger, O. Statistics for Management and Economics 4th Ed. Allyn and Bacon, Newton, MA, p 328.

Figure 1

Simulation of Confidence Intervals: Known Variance



*These six intervals do not contain the mean μ.
Of the 100 confidence intervals 94 contain the mean μ

Example 2

This example deals with the problem of estimating the mean of a normal population when the variance $sigma^2$ is unknown. The 100(1 - alpha)% confidence interval for the population mean mu when the population is normally distributed and the variance $sigma^2$ is unknown and estimated by the sample variance S2 is the interval bounded by the confidence limits

$$L = \bar{X} - t_{alpha/2,n-1}\ s/n^{1/2} \qquad (9)$$

$$R = \bar{X} + t_{alpha/2,n-1}\ s/n^{1/2} \qquad (10)$$

Using the sleep enhancing drug data that was used by Gosset, assume we know from past experience that the increase in sleep is normally distributed with some mean mu and the population variance is estimated by $\mathbf{S^2} = 1.513$ so s = 1.23. The 95% confidence interval for the sleep enhancing drug data is

We are 95% confident that the mean mu increase in sleep expected when using the drug lies between .70 and 2.46 hours.

$$L = \bar{X} - t_{alpha/2,n-1}\, s/n^{1/2} = 1.58 - (2.262)(1.23/10^{1/2}) = .70 \qquad (11)$$

$$R = \bar{X} + t_{alpha/2,n-1}\, s/n^{1/2} = 1.58 + (2.262)(1.23)/10^{1/2} = 2.46. \qquad (12)$$

The meaning of our being 95% confident is shown in the results of a simulation. We took 100 different random samples of size n = 10 each from a normal population with a mean mu and standard deviation sigma = (1.66) 1/2. One hundred sample means, X's, sample standard deviations, s's, and corresponding confidence intervals were then computed, and the results are presented in Figure 2. The mean mu was contained in 95 of the 100 intervals, as shown in the figure. This result conforms to our expectation that about 95 of our 100 intervals should encompass the mean mu.

Figure 2

Simulation of Confidence Intervals: Unknown Variance

*These five intervals do not contain the mean μ.
Of the 100 confidence intervals, 95 contain the mean μ.

Source: Watson, C., Billingsley, P., Croft, J., Huntsberger, D. Statistics for Management and Economics, 4th Ed. Allyn and Bacon, Newton, MA, p 335.

Notice that the lengths of the intervals in Figure 2 are different, in contrast to the intervals depicted in Figure 1. The lengths of the 100 intervals are different because of the different standard deviations s's for the samples. Both the sample means X's and the sample standard deviations s's are contributing to the variability among the intervals in Figure 2; whereas, the sample mean is the only variable that changes in Figure 1.

The simulation is performed as an experiential exercise to create new supplementary teaching materials for clarifying the concepts upon which decision making via control charts is based. Designed to fortify and deepen learning, the figures generated via the simulation provide visual support to facilitate interpretation of the control limits concept---a complicated and difficult one for many students new to the field of quality control.

The use of X charts for controlling a production process is roughly equivalent to performing a series of hypothesis tests (DelMar & Sheldon, 1988). Each time a sample is taken, a decision is made as to whether or not the process average or centerline is equal to the average stated on the X chart. This decision is based on whether or not the sample mean falls outside the control limits on the X chart. The control limits represent a decision rule that is inherently connected to the elementary aspects of confidence interval estimation.

The basic idea behind a control chart is that a set of sample statistics will have a distribution if only usual process variability influences the quantities. This distribution will have a mean and a standard deviation. Unless the distribution is extremely nonnormal, then relatively few points will be outside the range of the mean plus or minus 3 standard deviations. Thus we can use this fact to set up control limits. The Upper Control Limit (UCL) is set about 3 standard deviations above the distribution mean and the Lower Control Limit (LCL), about 3 standard deviations below the distribution mean.

The concepts of statistical processes and process control can also be introduced by using the results shown in figures 1 and 2. For example, statistical quality control limits could be placed on the figures at plus and minus three standard errors and analyses of runs for the sample means could be discussed to show that the processes are under control. For example, consider Figure 3, a control chart for a process mean X for electrical charge for memory chips. The X chant was constructed by plotting the subgroup means X j's, the central line at X-bar =10.05, and the upper and lower control limits, 8.648 and 11.45, for 12 hours. Note that the sample mean for the eleventh hour is outside the upper control limit, so the process is said to be out of control. Likewise, if Figures 1 and 2 are rotated 90 degrees and the sample means are connected, the figures would be quite similar in appearance to control charts. Thus, the figures created by the simulations show what the results of sampling look like when you are dealing with a stable process. In other words, constructing control charts is just another application of sampling and confidence intervals except the confidence limits and control limits are not set in the same way.

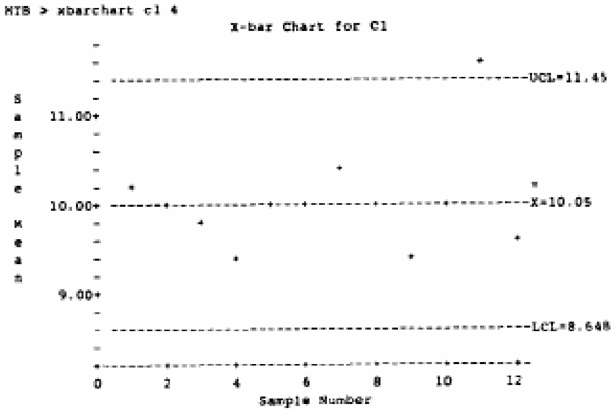
Source: Watson, C., Billingsley, p., Croft, J., Huntsberger, D. Statistics for Management and Economics, 4th Ed. Allyn and Bacon, Newton, MA, p 815.

Figure 3

X-bar Chart for C1

```
MTB > set c1
DATA>  9.9   9.9  11.0   9.8
DATA>  9.1   9.8   9.9  11.2
DATA>  9.6   9.4  10.7   9.9
DATA> 11.4   9.4   9.2   7.9
DATA>  9.9  10.6   9.6  10.0
DATA> 10.3   9.8   9.7   9.9
DATA> 11.2  11.1   9.6  10.0
DATA> 10.5   9.5   9.6  10.6
DATA>  8.5  10.3   8.3  10.2
DATA> 11.0   9.8  10.4   8.9
DATA> 10.5  12.2  10.3  13.3
DATA>  7.8  10.6  10.1  10.1
DATA> end

MTB > xbarchart c1 4
```



DISCUSSION

Understanding the concepts which underlie the setting of control chart limits is a problem encountered in discussing and teaching quality control. Often the students have little intuitive understanding of the meaning of the limits and the relationship of plotting (location) the sample means and how this procedure is associated with sampling and confidence intervals. The concept of process control can be lost in the maze of mechanical procedures, constants, and runs rules.

The formulation of control limits uses the notion of confidence in etimation. Our assumption in determining the character of our pedagogical method is that an appropriate groundwork for the study of control charts is a substantial amount of careful instruction in the area of confidence interval estimation. We adopted the position that to concentrate on relatively simple concepts of interval estimation before turning attention to the details of analysis and application of control charts makes good sense and is preferable in the long run to the more traditional mechanical approach.

By demonstrating the concept of confidence interval estimation, the results of the simulations provide sufficient background material and bridge the gap between theory and practice in the area of control charting. The plots show the results of the simulations in a compact form that makes the process of simulation objectively real and concrete. Furthermore, the figures lay the groundwork for the connection between confidence intervals, control limits, and process analysis.

The use of these innovative figures, which are generated by simulations and visually enhanced by computer, permits the student to steadily progress from aspects of confidence interval estimation in a simple context to relatively advanced phases of application in quality control. An underlying assumption here is that it is pedagogically preferable to pursue the joint development of intuition and rigor rather than to treat them separately.

Table 1

Additional Hours of Sleep Gained by using the Drug

Patient	Increase	Patient	Increase
1	1.2	6	1.0
2	2.4	7	1.8
3	1.3	8	0.8
4	1.3	9	4.6
5	0.0	10	1.4

X = 1.58 hours

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