Developments in Business Simulation & Experiential Exercises, Volume 16, 1989

THE PRODUCTION FRONTIER MODELING PRODUCTION IN COMPUTERIZED BUSINESS SIMULATIONS

Steven C. Gold
Thomas F. Pray

ABSTRACT

This paper presents a system of equations that can be used for modeling the production function in computerized business simulations. A review of a sample of contemporary business games demonstrates that most designers have production algorithms that are incomplete or inconsistent with the theory of production. The production system presented in the paper uses a continuous multiplicative functional form which is flexible and allows the designer to specify and control for: diminishing returns, points of inflection, output elasticities, and returns to scale. The complete production system consists of fifteen equations in which the first nine define the production frontier (i.e., outside boundary) and six optional functions allow the designer to incorporate input inefficiencies and/or improvements into the model. A numerical example demonstrates use of the frontier.

INTRODUCTION

The production function defines the technology of the business and the relationship between the inputs used by the business and the quantity of goods or services provided, the specification of this relationship should play an important role in business or management simulations. The costs of doing business and profits are directly related to the production function. The production function will influence the firm's optimal mix of inputs, inventory policy, capital utilization, and capital investment.

Business and management simulations are modeled to represent the "real world" firm. Participants in a simulation should gain insights into the workings of the real world through involvement with the simulation. As a result, it is necessary that the functions and algorithms within the simulation be consistent with the economic theory of production. Although the economic theory of production is well known, the task of incorporating the theoretical properties in to a computerized business simulation is not straightforward. In this regard, a paper by Kenneth Goosen (1986) stated:

"A major problem in the development of functional relationships is the creation of relationships that (1) are supported by theory, and (2) have minimum and maximum values or inflection points. Very little research has been published concerning the development of functional equations for business games. ... Satisfactory mathematical equations that have inflection points or maximum and minimum values at the desired points over a desired range of values are difficult to develop. In many cases equations that appear suitable only give desired results over a limited range of values."

The intent of this paper is to address these issues and to provide a theoretically sound structure for developing functional relationships of the production process in computerized business simulations.

PURPOSE

The paper has five goals:
(1) to summarize the properties of production theory, in both the short-run and long-run, that are most relevant to the modeling of business and management simulations;

(2) to review some of the ways in which the production function has been modeled in contemporary business and management simulations;

(3) to develop a production system that embodies the following key theoretical production properties: a multiplicative functional form, where the marginal product of each input depends on the level of the other inputs, points of inflection for each input, variable output elasticities, and increasing and decreasing returns to scale;

(4) to detail the equations derived from the production system that are needed to define the parameters of the system. The parameters are based on the designer's apriori specifications relating to the desired elasticities and inflection points;

(5) to demonstrate how the production frontier works with a numerical example.

PRODUCTION THEORY

A firm's production function is a mathematical expression describing the transformation of inputs into outputs. The concept of a production function is general and is defined with respect to a given technology. If a firm produces a single product or service, say Q, with "n" inputs, say X1 to Xn, we may write the production function as:

$$Q = f(X1,X2,X3,...Xn)$$

where: Q = quantity produced
Xn = the quantity of the nth input

For modeling purposes, this theoretical production function should be considered as the "production frontier" for it describes the maximum obtainable output (Q) that could be produced from the most efficient utilization of inputs (Xn). The frontier itself does not consider input waste or inefficiencies that often occur in the real world.

Short-Run Properties

The theoretical relationship between inputs and outputs, when there are fixed factors, may be characterized succinctly by a set of graphs for: Total Product, Average Product, and Marginal Product.

Figure 1
The Production Frontier

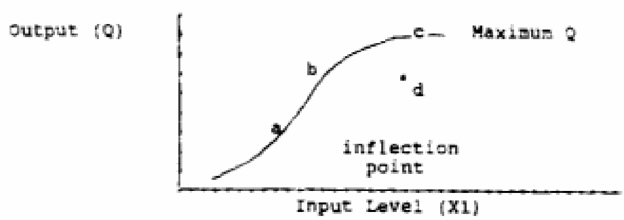
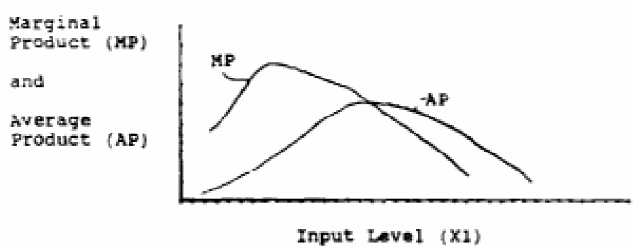


Figure 2
Marginal and Average Product



The Total Product (0), or the frontier, is "S" shaped, at first rising at an increasing rate until point A and then rising at a decreasing rate until point C. After point C, the maximum total product, output falls. The rate of increase or decrease in the slope of the function depends on the type of product; and would change as technology changes. The Total Product curve shifts up as the level of the fixed inputs increases. In Figure 1, if the firm was not operating in the most efficient manner it might well be fully within the frontier, say at point "d".

The Marginal Product (MP) is the first derivative of the Total Product (dQ/dX) and is "U" shaped (with maximum paint). Marginal Product rises until the inflection point of the Total Product and then declines, intersecting the horizontal axis at the point of maximum total product, point C. The MP depends on the level of X1 as well as the other factors of production, that is, MP = f(X1, X2 ,...., Xn)

The Average Product (a/X) is the total product divided by the input level and is also "U" shaped (with a maximum point) but does not intersect the horizontal axis. The average product rises as long as the marginal product exceeds the average product. The marginal product equals the average product when the average product is at its maximum level. The maximum point on the AP curve is to the right of the MP's maximum. The average product declines when the marginal product is below the average product. The average product approaches the horizontal axis asymptotically but does not intersect.

Long-Pun Properties

All inputs are variable in the long-run. Of primary interest is the property of returns to scale. Returns to scale measures the percent increase in output due to a percent increase in all inputs. Returns to scale depends on the output elasticities of each input.

The Output Elasticity (Ex) is the percent change in output owing to a percent change in one input, say x. Mathematically, the output elasticity is the marginal product divided by the average product. The output elasticity of input X1 may be expressed as:

$$El = (dQ/dX1) / (Q/X1) = MP / AP$$

The output elasticity is positively related to the marginal product but inversely related to the average product. Returns to scale (E) is the sum of all the output elasticities, such that:

$$E = El + E2 + E3 + ... + En$$

where En is the output elasticity of the nth input.

Increasing returns to scale occur when the percent increase in output is greater than the percent increase in all inputs, that is:

$$El + E2 + E3 + ... + En > 1.0$$

Constant returns to scale occur when the percent increase in output is equivalent to the percent increase in all inputs:

$$E1 + E2 + E3 + ... + En = 1.0$$

Decreasing returns to scale occur when the percent increase in output is less than the percent increase in all inputs, that is:

$$E1 + E2 + E3 + ... + En < 1.0$$

The specification of returns to scale depends on the product and industry. Electric power and automobile manufacturing are examples of industries with relatively high returns to scale compared to service industries which have relatively low returns to scale. Increasing returns to scale are followed by constant returns to scale and, eventually, decreasing returns.

REVIEW OF CONTEMPORARY SIMULATIONS

Eight simulations were selected to study how the designers modeled the production function. The simulations investigated were composed of five business policy/management decision-making games [Jensen and Cherrington (1984), Mills and McDowell (1985), Keys and Leftwich (1985); Pray and Strang (1980), Scott and Stickland (1985)); one marketing game (Faria, Nulsen, and Roussos (1984)]; a production-operations game [Pray, Strang, Gold, and Burlingame (1984)), and an international game (Edge, Keys, and Remus (1985)]. A brief summary of each of their production functions is presented below:

Two games emphasized production-related decisions: Business Management Laboratory (BML) and DECIDE-P/ON. Both BML and DECIDE- have the firms producing two products, using two raw materials and have two phases in the production process. Raw materials are used in a constant ratio to output, but DECIDE-P/OM has raw material waste considerations tied to quality of the suppliers. Both have plant capacity expressed in terms of labor and a constant labor/output ratio. In the labor decision is automatically made through the production schedule and DECIDE-P/OM treats labor as a decision variable. Both permit capacity expansion in terms of a fixed dollar to capacity increase ratio - with time lags. In DECIDE-P/ON there is a downtime percentage (non productive labor) which is influenced by production level, maintenance expenditures, uncertainty, and capital intensification policy. In there is an engineering study decision variable which may lower labor costs, but does not impact capacity, and a maintenance decision which impacts the decline of capacity. Both games have shift considerations with overtime premiums.

The Multinational Management Game, The Executive Simulation and COMPLETE: A DYNAMIC MARKETING SIMULATION place more emphasis on marketing than the production function. In these games, there are multiple outputs and there are no raw material decisions nor labor decisions. In COMPETE, production is tied directly to the company's forecast. It is assumed that each company has sufficient capacity to produce all the goods that are needed and plant expansion is not considered. In the Multinational Game and the Executive Simulation, plant size can be expanded or contracted by a fixed dollar to plant increase or decrease ratio. Overtime is permitted.

DECIDE and MICROMATIC may have three inputs (labor, material, and capital) and one output. Both have raw material decisions subject to waste or loss of material. Both permit plant expansion but with fixed-dollar- to-increase-capacity ratio. Both games have labor decisions with a fixed input/output ratio and overtime can be scheduled in both. DECIDE has downtime applied to labor and MICROMATIC has a labor turnover factor.

The BUSINESS GAME increases the number of decision variables with time. In period five, teams may opt to modernize their plant and reduce their labor costs, but capacity does not change. Normally raw materials are automatically purchased by the algorithm, except in period seven, where firms may consider quantity discounts and purchase additional materials. In the eighth period, the firm is given the opportunity to expand its plant, with larger plants costing proportionately more than smaller plants.

Concerns and Shortcomings

The sampled simulations vary greatly in their level of complexity, overall educational intent, and in the way they transform inputs into outputs. As would be expected, the more operations-oriented games have more "production-related" types of decisions, whereas the international and marketing simulations place little emphasis on the production side. None of the games, however, discuss important short-run production concerns such as concern for diminishing returns to labor. Most simulations that permit capital expansion has fixed dollar ratios for plant expansion. This "fixed ratio" approach imposes constant returns to scale on all firms over any time horizon.

Care needs to be taken in the design of the production function. Players of simulations should be exposed to both diminishing return and economies-of-scale issues. Poorly designed production functions may lead to single optimal strategies such as: "Expand the plant as quickly as possible in the early stages of the game and no one will ever catch you!" or "Don't bother with plant investment it will never pay off" If the model does not have diminishing returns accurately reflected to labor, players may end up *with* unrealistic labor schedules for a fixed plant size.

Some designers have used increasing or decreasing factor costs to address scale or diminishing return issues. While this may seem to be adequate, theory purports that both diminishing returns and economies of scale concepts stem from the production function, and then exhibit themselves in the cost structure. For this reason we recommend incorporating both of these short run and long run concepts into the production frontier.

A RECOMMENDED PRODUCTION SYSTEM

In this section functional form for the frontier is presented that is consistent with the economic theory of production and is flexible. The form is multiplicative in nature with variable output elasticities and interdependent marginal products. For illustrative purposes the functional form is shown with only 3 inputs but may be easily generalized to any number of inputs.

Production Frontier Function

$$Q = a1\, U^{(a2 - a3U)}\, S^{(a4 - a5S)}\, K^{(a6 - a7K)} \quad (1)$$

where: Q = maximum output
U = unskilled labor
S = skilled labor
K = capital
ai = parameters, i = 1...*7*

The output elasticities associated with each independent variable or input are:

Output Elasticities

Eu = a2 - a3 U (1 + lnU) ; unskilled labor (2)

Es = a4 - as S(1 + lns) ; skilled labor (3)

Ek = a6 - a7 K(1 + lnK) ; capital (4)

where: "lnX" denotes the natural logarithm of input X.

At the inflection point (or point of diminishing returns) the second derivative of the function is zero. The relationship between the output elasticities and the level of the independent variables (inputs) at the inflection point is:

Inflection Points

$(Eu)^2$ = a2 + a3 U ; unskilled labor (5)

$(Es)^2$ = a4 + a5 S ; skilled labor (6)

$(Ek)^2$ = a6 + a7 K ; capital (7)

Returns to scale (E) is equal to the sum of the output elasticities for unskilled labor, skilled labor, and capital:

Returns to Scale

$$E = Eu + Es + Ek \quad (8)$$

Finally, an output constraint should be imposed to guarantee stability in the simulation:

Output Constraint

$$Q > 0 \quad \text{and} \quad Q < Qmax \quad (9)$$

The quantity produced must be greater than or equal to zero; and less than or equal to the maximum feasible level of output per period, defined as Qmax.

Input Inefficiencies and/or Improvements

Equations 1-9, the "core" equations of the system, are used to design and determine the production frontier. Designers, however, may want to place a greater emphasis on the operations side and include other decision variables into their algorithm including:
preventative maintenance, quality control variables, training considerations, multiple shifts, capital intensification, etc. To make the game more realistic it is often desirable to introduce inefficiencies into the production system so that actual production occurs within, and not on the frontier. Three different examples of inefficiencies, incorporating other decision variables, are briefly discussed below. These include: nonproductive labor, raw material waste, and capital intensification issues.

NONPRODUCTIVE LABOR

Labor inefficiency can be introduced as a percentage and then used to reduce the scheduled labor input. By reducing the labor input, the firm will operate within the frontier, but still have the frontier as a boundary. Labor inefficiencies may be a result of inadequate training, instructor control, or just random elements. Equation 10 represents the nonproductive labor percentage (NPL*).

NPL% f (z1, Z2, Z3, Z4) (10)

where: NPL% = the nonproductive labor percentage

Z1 = a certain base percentage
Z2 = a multiplier factor based on training
Z3 = a multiplier factor based on instructor control
Z4 = a multiplier based on a random number

A firm may have a fairly constant labor downtime percentage (NPL%) based on history (Z1), but this percentage could either be decreased or increased by effective or ineffective training programs (Z2) respectively. It could also change because of instructor control (Z3), say to simulate a work slowdown, or just fluctuate mildly because of random elements (Z4). If such factors are included in the simulation then the labor input would be less than actually scheduled. Equation 11 demonstrates how the unskilled labor input In Equation 1 would be reduced because of the nonproductive percentage. The simulation team would be charged for (SX1) labor but only have (U) labor hours used in the production frontier calculations.

U = SX1 * (1-NPL%) (11)

where: U = the effective labor input
SX1 = scheduled labor usage
NPL% = the waste percentage

WASTE ON RAW MATERIAL INPUTS

Waste of materials is often a real world consideration, with a certain level of waste inherent to the production process. Reasons for waste include: poor input quality control procedures by the buyer, supplier problems, human error, or just bad luck. Waste of raw materials can be readily incorporated into a production system. A raw material waste percentage (WRM%) is presented, in functional form, in Equation 12.

WRN% f(Y1,Y2, Y3, Y4, Y5) (12)

where:

WRN%	=	the raw material waste percentage
Y1	=	a certain base percentage level
Y2	=	a multiplier factor based on quality control decisions
13	=	a multiplier factor based on instructor control
Y4	=	a multiplier factor based on research and development or engineering studies
Y5	=	a random number

As an example, a firm may have an inherent 10% waste (Y1) on their raw materials, but this percentage could increase to say 15% because of inadequate quality control decisions (Y2), instructor control (13), or just random or bad luck considerations (Y5). The designer could also allow for a reduction in the waste percentage, say from 10% to 5%, by allowing for a reduction in the base (Y1) by well thought out and effective quality control decisions (Y2) or by other variables such as R&D breakthroughs or engineering study improvements (14). If such factors are included in the simulation then raw material input would be less than actually scheduled as described in Equation 13.

X2 SX2 * (1-WRM%) (13)

where: X2 the effective raw material input sx2 = scheduled raw material usage
wRM% = the waste percentage

CAPITAL INTENSIFICATION ISSUES

In much the same fashion, capital inputs may be altered to impact productivity. If a firm fails to replace worn-out equipment, has an inadequate preventative maintenance plan, or expands plant too quickly, capital productivity should decrease. Capital inputs, on the other hand, may increase productivity as the firm replaces machinery with new technology embodying the latest improvements. Equation 14 presents a possible functional approach with a capital downtime percentage (CD%).

CD% = f(P1, P2, P3, P4, P5) (14)

where: CD% the capital input downtime percentage

P1		a certain base percentage level
P2	=	a multiplier factor based on replacement schedules.
P3	=	a multiplier factor based on capital intensification
P4	=	a multiplier factor based on preventative maintenance
P5	=	a random number

The model could have the base percentage (P1) influenced both positively and negatively by capital intensification. A well thought out capital and maintenance plan could reduce the downtime percentage significantly and place the firm near or on the frontier. If Equation 14 is implemented, the capital input in the frontier (Equation 1) would be altered in the same fashion as the other inputs have been.

K = sx3 * (l-CD%) (15)

where K = the effective capital input
sx3 = scheduled capital usage
CD% = the capital downtime percentage

Equations 10-15demonstrate how inefficiencies (or improvements) to input may be modeled. These were left in functional form because we consider them "optional" - to be used if the designer opts to have more of the production orientation to the simulation. But even if the game is not a production- oriented game, Equations 1-9, which define the production frontier, are highly recommend for all business simulations. The frontier is consistent with theory and can be specified apriori to meet the designers need. An example of defining the parameter and modeling the frontier is presented below.

SOLVING FOR THE PARAMETERS OF THE FRONTIER FUNCTION
An Example

Suppose a simulation designer wants to the following production characteristics:

Input	Output Elasticities	Inflection Points	
U	1.25	1000	hours unskilled labor/week
s	1.00	500	hours skilled labor/week
K	1.75	100	units capital/week

Scale E = 4.00 Q = 10000 units produced/week

At the point of diminishing returns (inflection point) the desired output elasticities for unskilled labor, skilled labor, and capital are respectively: 1.25, 1.00, and 1.75. The desired inflection points correspond to 1000 hours of unskilled labor, 500 hours of skilled labor, and 100 units of capital. The level of production at the inflection point is 10,000 units per week. Returns to scale (E) is 4.00, indicating a 1 percent increase in all factors will increase production by 4 percent. Consequently the production process is characterized by "increasing" returns to scale.

To solve for the parameters of the system, first substitute the given data into the output elasticity and inflection point equations (2 - 7) as illustrated:

Output Elasticities

1.25 = a2- a3 1000 (1 + ln1000); unskilled labor
1.00 = a4 - a5 500 (1 + ln500) ;skilled labor
1.75 = a6 - a7 100 (1 + ln100) ;capital

Inflection Points

$(1.25)^2 =$ a2 + a3 1000 ; unskilled labor

$(1.00)^2 =$ a4 + a5 500 ; skilled labor

$(1.75)^2 =$ a6 + a7 100 ; capital

Given 6 equations (above) and 6 unknowns (a2 to a7) it is possible to solve the system simultaneously to obtain the following results:

a2 = 1.5274182
a3 = 0.0000351
a4 = 1.0000000
a5 = 0.0000000
a6 = 2.8637920
a7 = 0.0019871

Output Level and Production Equation

To solve for parameter al and obtain the desired production level of 10,000 units per week, given the input mix specified previously, substitute the parameter values (a2 to a7) into the production function (equation 1) as shown:

$$Q = al\ U^{(1.53 - 0.000035\,U)}\ S^{(1.0)}\ K^{(2.86 - 0.01987\,K)}$$

given: Q = 10000;
U = 1000;
S = 500;
K= 100.

Substituting the values for output and input (Q, U, S, K) into the production equations and solving for al, we get:

$$al = 3.1183 \times 10^{-9}$$

SIMULATING THE DERIVED PRODUCTION FUNCTION

The production function derived in the numerical example should display all the theoretical properties described earlier. To illustrate the properties of the function and to show how it behaves when input variables are changed, the function will be simulated. The simulation will display two cases: (1) the short-run production function, by increasing unskilled labor while holding the other inputs fixed; and C2) the long-run production function, by increasing all inputs simultaneously.

Table 1: The Short-Run Production Function

Unskilled Labor	Units Produced	Average Product	Discrete Marginal Elasticity	Output
600	5,104	8.51	11.40	1.37
700	6,292	8.99	11.88	1.34
800	7,512	9.39	12.20	1.31
900	8,750	9.73	12.38	1.28
1000	10,000	10.00	12.50	1.25
1100	11,249	10.23	12.49	1.22
1200	12,490	10.63	12.41	1.19
1300	13,717	10.55	12.27	1.15
1400	14,925	10.66	12.08	1.12
1500	16,108	10.74	11.83	1.09
1600	17,264	10.79	11.56	1.06
1700	18,388	10.80	11.23	1.02
1800	19,478	10.82	10.90	0.99
1900	20,531	10.80	10.53	0.96
2000	21,546	10.77	10.16	0.92

Table 2: The Long-Run Production Function

Unskilled Labor	Skilled Labor	Capital	Units Produced	Returns to Scale
826	413	83	4572	4.27
909	455	91	6800	4.15
1000	500	100	10000	4.00
1100	550	110	14529	3.84
1210	605	121	20764	3.66
1331	666	133	29128	3.45
1464	732	146	39920	3.22
1611	805	161	58802	2.98
1771	886	177	70129	2.68
1948	974	195	89736	2.35
2143	1071	214	110627	2.03

The results of the simulation are reported in Tables 1 and 2.

The short-run function is consistent with the theory of production as outlined earlier. Output increases at an increasing rate until the inflection point at 1000 labor hours and then increases at a decreasing rate. Marginal product is maximized at the inflection point while average product continues to rise until 1800 labor hours and thereafter declines. The output elasticity is a decreasing function with respect to labor and is equal to 1.25 at the inflection point as specified in the example.

The long-run function is also consistent with the theory of production. Returns to scale is 4.0 at the inflection point of the function (as specified in the example) and continues to decline with increases in output. At higher levels of output, increasing returns to scale will eventually change to constant and then decreasing returns to scale with an elasticity less than 1.0.

CONCLUSION

An approach for modeling production that is theoretically consistent has been presented. The recommended approach uses a continuous multiplicative functional form which has the flexibility to model inflection points with corresponding minimum and maximum values. A numerical example was provided to illustrate how the system can be solved and used to model a given production technology. A simulation of the system showed it to be stable and consistent with apriori expectations.

Once the frontier is defined, Equations 10-15 may be used to introduce inefficiencies to inputs. These input inefficiencies would prevent the firm from producing on the frontier, if poor or ill-informed decisions were implemented. But by applying the inefficiencies to the inputs, and not the output, the frontier system remains intact and will behave in a manner consistent with theory and the designer's expectations.

REFERENCES

Edge, A.; Keys, B.; and Remus, W. (1985) THE MULTINATIONAL MANAGEMENT GAME, Dallas, Texas: Business Publications, Inc.

Faria, A.; Nulsen, R.; and Roussos, D. (1984) COMPETE: A DYNAMIC MARKETING SIMULATION, Dallas, Texas: Business Publications, Inc.

Gold, S. and Pray, T., "Modeling Market and Firm Level Demand Functions in Simulation & Games: An International Journal of Theory. Design, and Research, Vol. 15, No. 3, Sept. 1984. pp. 183-187.

Goosen, K. (1986) "An Interpolation Approach to Developing Mathematical Functions for Business Simulations" in Developments in Business Simulation and Experimental Learning, Vol. 13, 1986, pp. 248-255.

Jensen, R. and Cherrington, D. (1984) THE BUSINESS MANAGEMENT LABORATORY. Dallas, Texas: Business Publications, Inc.

Keys, B. and Leftwich, H. (1985) EXECUTIVE SIMULATION, New York, New York: Kendall Hunt.

Mills, L. and McDowell, (1985), THE BUSINESS GAME, Boston, Massachusetts: Little Brown & Company.

Pray, T. and Strang, D. (1980), DECIDE, New York, New York: Random House.

Pray, T.; Strang, D.; Gold, S.; and Burlingame, D. (1984), DECIDE-P/OM, New York, New York: Random House.

Scott, T. and Stickland, A. (1985), MICROMATIC A MANAGEMENT SIMULATION, Boston, Massachusetts: Houghton Mifflin Company.