AN INTERPOLATION APPROACH TO DEVELOPING MATHEMATICAL FUNCTIONS FOR BUSINESS SIMULATIONS

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ABSTRACT

Simulation designers are often required to define the precise nature of functional relationships with only vague guidelines from theory or practice. This paper presents a methodology for the easy implementation of self-designed functional relationships. The approach presented does not require any special parameters or exponents; yet it is capable of generating results almost identical to sophisticated curvilinear functions. The method described in this paper relies on interpolation as the means of obtaining dependent variable values from a graphical representation of a functional relationship.

INTRODUCTION

The design of general management business and functional area simulations involves writing scenarios, selecting decisions, formatting financial statements, creating fictional historical data, and developing mathematical functions. The results of all these activities except the mathematical equations are visible through description in the student simulation manual. The scenario with its descriptive details gives the simulation a static appearance of reality; however; it is the set of mathematical equations embedded in a computer program that provides the dynamics of simulated realism.

Game designers for the most part have relied upon mathematical equations to create quantitative relationships among business simulation variables. A major problem with mathematically defined functional relationship equations is that these equations tend to control the magnitude and breadth of simulation parameters. In some cases the equations have given undesirable results at certain decision points and the game designer has been forced to create artificial limits to restrict certain decisions. Analysis by Gold and Pray of currently used games showed that game designers have experienced considerable difficulty in developing equations that give the desired results over a wide range of decision values. /1/

A game normally contains a number of different functional relationships. For many relationships the exact functional behavior is not prescribed by theory or indicated by empirical data. For relationships involving credit, wage rates, commission rates, salaries, etc. equations that give the results desired by the game designer or required by theory may not exist or be difficult to develop. Also, in instances where theoretical functional relationships have been developed, the mathematical complexity of the equations may be beyond the expertise of the game designer to implement.

Because game designers are often required to define the precise nature of functional relationships with only vague guidelines from theory, an easy method of implementing self-designed functional relationships is desirable. This paper presents such a methodology. The approach presented does not require any special parameters or exponents; yet it is capable of generating results almost identical to sophisticated curvilinear functions.

The method presented in this paper requires that the game designer first graphically define the desired functional relationship. The graphical representation of the functional relationship may have multiple inflection points and multiple minimum and maximum points, if this is desired by the game designer. The method described in this paper relies on interpolation as the means for obtaining dependent variable values for every possible independent variable value. The technique does not in any way prescribe the nature of the functional relationship. However, the methodology presented can duplicate and improve the results of mathematical functional relationships that tend to deteriorate at certain levels of activity. The advantage of the interpolation approach presented is that it gives the game designer complete control output over the entire prescribed range of sales and production activity.

BASIC THEORY OF DEVELOPING FUNCTIONAL RELATIONSHIPS

Various types of equations may be found within the simulation computer program. They vary from the simple to the complex. While some are linear in nature, others may be curvilinear. Some are fairly complex equations which serve to determine units manufactured and units sold. Others simply serve as counters to determine balances for financial statements. Of the various types, the functional, equations are those most critical to simulating reality of economic forces.

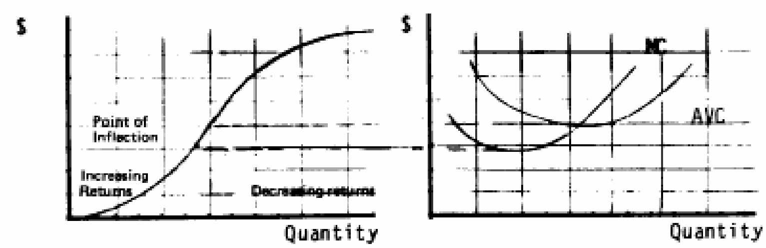
Functional equations are equations in which the independent variable (usually a decision value) is an input into a function which determines the value of the dependent variable. The resulting dependent variable value is either an output for financial statements or an input value for other functional equations. The existence of functional equations creates opportunities for students to make optimal decisions. Without these equations, the results of game play would have no relationship to reality. Increasing values for price or advertising, e.g., would have no adverse consequences.

In an earlier paper, Goosen /2/ established the importance of functional relationships. "If a business simulation is to be realistic, the same general relationships found in the real world must be found within the simulation. In order for a simulation to have the needed air of realism and dynamism, mathematical functions must be created to simulate these economic forces." Examples of decisions that require functional relationships with points of inflection or minimum and maximum values are price, advertising, research and development, wage rate, credit, salesmen commissions.

A major problem in the development of functional relationships is the creation of relationships that (1) are supported by theory, and (2) have minimum and maximum values or inflection points. Very little research has been published concerning the development of functional equations for business games. A notable exception is the work done by Gold and Pray /3/. Gold and Pray have done an outstanding job in several papers presented at ABSEL in setting forth procedures for the development of market demand equations. Their approach represents a major advancement in business simulation modeling. Their work should be read and understood by serious game designers. However, their model is an application to a specific area, market demand. The interpolation approach in this paper is general and can be applied to any decision area of a simulation.

An ideal functional relationship is one in which for a range of desired decision values (the independent variable) a maximum or minimum value exists or contains points of inflection. In some functions, inflection points are necessary in order to simulate real world behavior. Mathematically, an inflection point is the point in a line where there is a change in the curvature from concave to convex or conversely. For total cost curves this means that the related marginal cost and average cost curves have minimum values. A graphical example of a total cost curve and the associated marginal cost and average cost curves is shown in Figure 1.

FIGURE 1
ILLUSTRATION OF TOTAL COST CURVE WITH AN INFLECTION POINT



Points of inflection are desirable in business simulation functional relationships because they create increasing and decreasing returns. For example, this means that successive equal increases in advertising would have differing effects. At some point the additional cost of advertising would exceed the additional revenue.

AN INTERPOLATION TECHNIQUE FOR DEVELOPING MATHEMATICAL FUNCTIONS

Satisfactory mathematical equations that have inflection points or maximum and minimum values at the desired points over a desired range of values are difficult to develop. In many cases equations that appear suitable only give desired results over a limited range of values. Furthermore, in order to control the output values within the desired limits of economic activity, the creation of parameters (constants) is often necessary. A satisfactory functional relationship is one that (1) provides a realistic response to a decision value, and (2) produces realistic values over the entire range of sates and production activity.

The interpolation method covered in this paper gives the simulation designer control over desired output values over a specified range of activity. Although some of the elegance and symmetry of a true curvilinear functional equation containing inflection points is sacrificed, the results of the interpolation approach are not significantly different. Therefore, the simplicity of design and the ability to control desired results justifies the loss of mathematical elegance for mathematical manipulation.

The interpolation method to be described is a technique that attempts to approximate functional equations that have minimum or maximum values or that have points of inflections. Interpolation is an operation for determining a number or value not listed in a table or array of numerical values by deriving the missing value on the basis of proportions computed from the numbers that are listed in the table. The interpolation model for computing functional relationship values is described in Appendix A. A computer program for applying this model is presented in Appendix B.

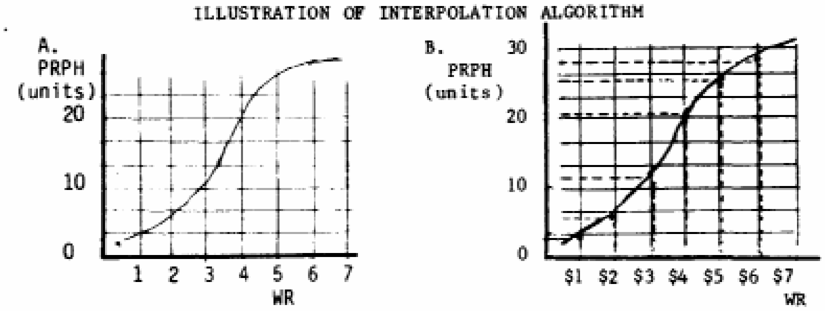
The interpolation method for developing mathematical functions for business simulations involves the following steps:

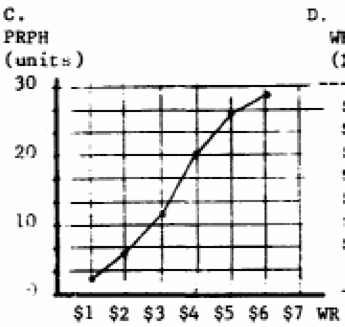
1. Prepare a graph having the desired calibrations and draw or sketch the desired curvilinear functional relationship.
2. Select points at regular intervals on the drawn functional relationship in order to prepare an interpolation schedule.
3. Prepare a schedule of interpolation values by measuring the coordinate values of the points where each linear line connects to another linear line. These ordered paired values may be called range values.
4. For any independent variable value not listed in the table of ordered pairs of values, compute the missing dependent value by using the interpolation equation presented in Appendix A.

Illustration

Assume that a functional relationship between the wage rate and production rate per hour is desired. Furthermore, assume that up to a given wage rate the returns are increasing followed by decreasing returns with increases in wages. To accomplish step 1, create a graph and draw the desired functional relationship as shown in Figure 2A. Then as illustrated in Figure 2B, determine points on the drawn functional relationship to serve as the basis for an interpolation schedule. The effect of this process is to create a linear approximation of the functional relationship as illustrated in Figure 2C. For step 3, identify the corresponding coordinate values of each point *(see* Figure 2B) and then prepare a table showing arrays for the X (wage rate) and Y (PRPH) values, as illustrated in Figure 2D. Values not explicitly shown in the table of X and *Y* values may be computed by interpolation, as shown in Figure 2E. The interpolation values were computed based on the interpolation equation presented and explained in Appendix A. The interpolation value, IV, for PRPH at a wage rate of $1.8 was computed as follows:

FIGURE 2
ILLUSTRATION OF INTERPOLATION ALGORITHM





D.

WR (X)	PRPH (Y)	Cost Per Unit
$1	2	$.50
$2	6	$.33
$3	12	$.25
$4	20	$.20
$5	25	$.20
$6	28	$.21
$7	30	$.24

E.

	WR (X)	PRPH (Y)
1	$1.0	2.0
*	$1.2	2.8
*	$1.4	3.6
*	$1.6	4.4
*	$1.8	5.2
2	$2.0	6.0

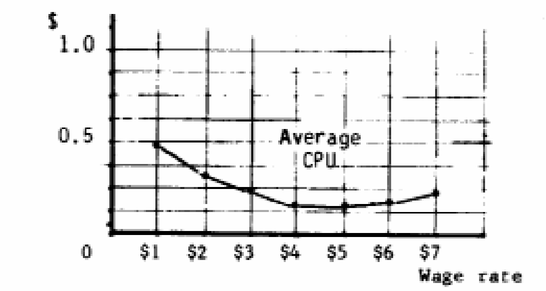
*Interpolated values

$$IV = Y1 + \left[\frac{X2 - WR}{X2 - X1} \cdot (Y2 - Y1)\right] = 2 + \left[\frac{2 - 1.8}{2 - 1} \cdot (6 - 2)\right] = 5.2$$

As long as decision for the wage rate remains in the $1 to $7 range, the PRPH for any value of the wage rate can be determined. The interpolation method, therefore, produces a set of dependent and independent variable values that can serve as substitutes for values generated from a true curvilinear functional, equation.

The interpolation schedule in Figure 2D does, in fact, contain an inflection point. Average cost per unit is minimized at a wage rate of $4 and $5.

Graphically, the interpolated average cost per unit schedule may be presented as follows:



AN INTERPOLATION MODEL FOR MARKET DEMAND

The interpolation technique just illustrated can be used to develop a satisfactory market demand model involving price and advertising as variables. A good market demand model involving a multiplicative type mathematical equation has been presented by Gold and Pray./3/ An interpolation-based market demand model which will give results similar to the Gold and Pray model will now be described. In the next section the results from the interpolated model will be compared to results computed by the Gold and Pray model using the same decision values. The purpose of this comparison is to demonstrate that the interpolating approach can achieve almost identical results without the need for parameters or exponents.

The Gold and Pray model is summarized in Figure 3. This model is described by Gold and Pray as a generalized multiplicative model; that is, a change in a multiplier in the equation does not change other values in the equation which serve as multiplicands. The equation for market demand which includes price, marketing (advertising), and research and development as independent variables is based on an exponential function in which the independent variables also appear as variables in the exponent. In the Gold and Pray equation, the exponents are linear equations containing price, advertising, and research and development as variables.

Gold and Pray defined P, R, and M as exponentially smoothed values; however, this assumption is not essential to the analysis that follows:

Following the procedure used by Gold and Pray in their paper, the equation for R & D will be dropped for illustrative purposes and the analysis will concentrate on price and advertising. The Gold and Pray demand model contains 5 parameters for price and marketing: g1, g2, g3, g4, g5. The values for these parameters are determined by the designer's specification concerning the elasticities associated with the demand. In their paper, Gold and Pray assumed the following elasticities for two levels of prices and marketing.

Price	Price Elasticity	Marketing ($)	Marketing Elasticity
$10.00	0.5	$ 50,000	3.0
$20.00	1.0	$150,000	1.0

FIGURE 3

THE GOLD/PRAY GENERALIZED MULTIPLICATIVE MARKET DEMAND MODEL

$$Q = g1\, P^{-(g2 + g3P)}\, M^{+(g4-g5M)}\, R^{+(g6-g7R)}$$

where: Q = The total market demand
P = Price
R = Research and development expenditures
M = Marketing expenditures
gi = parameters or constants (for i = 1,7)

Based on these assumptions, the following values for g1, g2,…..g5 were computed by Gold and Pray:

g1 = (2.34 x 10 -12) g2 = .15, g3 =.01 g4 = 3.88, g5 = .0000015

Given these parameters, Gold and Pray then computed the total market demand at different prices and different levels of advertising. The results of their calculations are presented in Figure 4.

FIGURE 4

EFFECT OF PRICE AND ADVERTISING ON DEMAND

Adv. = $50,000		Price = $10	
Price	Quantity Demanded	Marketing Expenditures	Quantity Demanded
$10	997,228	50,000	997,228
$12	906,603	70,000	1,567,199
$14	824,933	90,000	4,708,172
$16	750,791	110,000	7,045,763
$18	683,213	120,000	9,206,539
$20	621,492	150,000	10,910,114
$22	564,065	170,000	12,011,372
$24	513,468	190,000	12,483,417
$26	466,291	210,000	12,385,871
$28	423,172	230,000	11,828,334
$30	383,788	250,000	10,939,432
		260,000	10,410,487

These values generated by the Gold and Pray demand model, have been presented graphically in Figures 5 and 6.

At a price of $20, the elasticity of demand is 1. At this price sales is maximized at $12,429.845. At an advertising level of $190,000 market demand is maximized at 12,483,417 units. The Gold and Pray price/quantity demand curve is curvilinear and therefore not of the linear type most frequently illustrated in principles of economics textbooks.

Implicit in the Gold and Pray model is a functional relationship between advertising and market demand. This relationship is illustrated in Figure 6. Notice that Figure 6 shows the relationship between advertising and quantity to have a maximum value and that a point of inflection exists indicating increasing and decreasing returns.

An unique characteristic of the Gold and Pray Demand model is that when advertising is incremented, the percentage increase in market demand is constant at all levels of price. To demonstrate that the Gold and Pray model involves constant percentage in total marker demand changes at different levels of price, the Gold and Pray model was used to generate the data appearing in Figure 7. In Figure 7, total market demand was computed at advertising levels of $50,000, $90,000, $130,000 and $170,000. All percentage changes are computed with quantity at $50,000 of advertising as the base amount.

Notice that an increase in advertising from $50,000 to $130,000 had an 823% increase in total market demand at all levels of prices. This means that a change in advertising simply shifts the demand curve to the right with the slope of the line decreasing with each incremental increase in advertising. For each price, the percentage increase in quantity is the same.

FIGURE 5

GRAPH OF TOTAL DEMAND AT VARIOUS PRICES

Adv. = $50,000

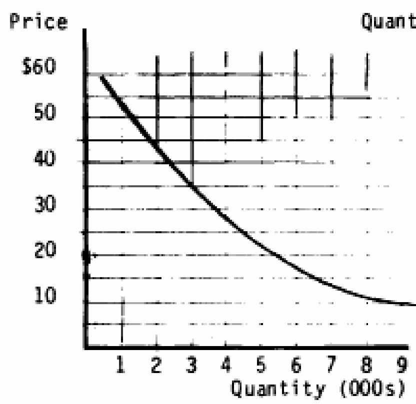
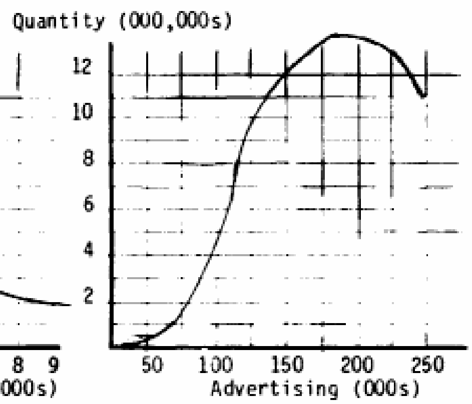


FIGURE 6

GRAPH OF TOTAL DEMAND AT VARIOUS LEVELS OF ADV.

Price = $10



The effect of advertising on total demand at advertising levels of $50,000 $130,000 and $170,000 is graphically illustrated in Figure 8. Notice that a change in advertising shifts the demand curve to the right such that a family of curves results.

The fact that the Gold and Pray model implicitly multiplies demand determined by different prices by a constant percentage for a given change in advertising means that the effect of marketing expenditures can be stated simply as percentage change rather than in terms of units. This fact is quire significant because the interpolation approach can be used to duplicate the results of the Gold and Pray model in a very simple and straight forward manner without the need for an exponential, model. Therefore, from an interpolation viewpoint, the equation for total market demand can be mathematically stated as:

$$TQ = PQ + PCa(PQ) + PCr(PQ) = PQ (1 + PCa + PCr)$$

where

TQ = total market demand
PQ = quantity due to price
PCa = percentage increase in demand due to adv.
PCr = percentage increase in demand due to R & D

The observation that a change in advertising affects quantity at all prices by an equal percentage applies to research and development since the function for research and development is identical to advertising. Therefore, if R & D is set to zero, then the equation becomes:

$$TQ = PQ * (1 + PCa)$$

PQ and PC are values that can be computed by interpolation from the table of values for price and advertising.

COMPARISON OF THE COLD/PRAY AND INTERPOLATION MODELS

To demonstrate that the Pray/Gold model and the interpolation model give comparable results, the above interpolation demand model will be used to compute total market demand at different prices. Advertising initially will be assumed to be $50,000.

Since the interpolation technique requires arrays of paired values, the range points to be used in the interpolation model are shown in Figure 9. The range points are derived from data generated by the Gold and Pray model. Since the range points are points on the demand line created by the Gold and Pray demand model, the interpolation model will produce identical market demand values at these points. However, the question to be answered is: For values not explicitly given in the interpolation table of values, how closely will the interpolated values correspond to values generated by the Gold and Pray demand model?

Based on the Gold and Pray model and the illustrative data used by them, the following tables of paired values can be listed as the range values for the following interpolation schedules.

The above interpolation schedules may be presented graphically as illustrated in Figure 10.

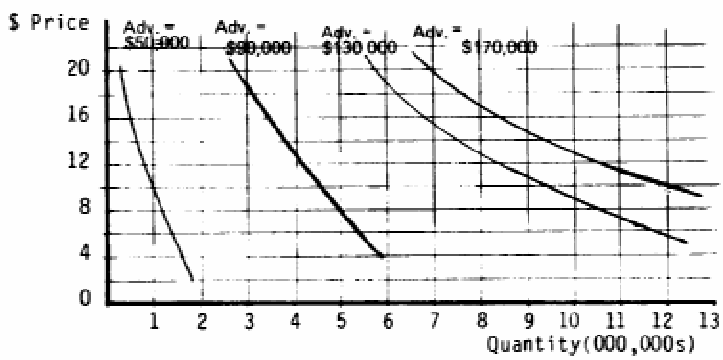
Suppose that price in a game competition ranges from $20 to $40 and that advertising is $110,000. What values for quantity would result through interpolation? Fig. 11 shows these amounts. Corresponding amounts are shown computed from Gold and Pray model.

The interpolation approach closely approximates the results of the multiplicative demand model of Gold and Pray. In fact, when the difference in demand is computed as percentage of the quantity derived from the Gold and Pray model, the highest percentage variation is less than 12%. A much higher degree of accuracy can

FIGURE 7
PERCENTAGE EFFECT ON TOTAL MARKET DEMAND OF CHANGES IN ADVERTISING

	Advertising						
Price	$50,000	$90,000	%C	$130,000	%C	$170,000	%C
$10	997,228	4,708,182	372%	9,206,519	823%	12,011,366	1104%
$12	906,003	4,280,300	372%	8,369,827	823%	10,919,769	1104%
$14	824,933	3,894,718	372%	7,615,848	823%	9,936,083	1104%
$16	750,791	3,544,672	372%	6,931,356	823%	8,229,102	1104%
$18	683,213	3,225,621	372%	6,307,474	823%	7,485,690	1104%
$20	621,492	2,934,220	372%	5,737,661	823%	6,806,064	1104%

FIGURE 8
GRAPH OF EFFECT OF CHANGE IN ADVERTISING ON DEMAND CURVE



be achieved by decreasing the distance between the range values. If the interpolation had been between prices of $40 and $30 rather than $40 and $20 the maximum percentage variation would have been less than 2% at all levels of price.

FIGURE 9
INTERPOLATION SCHEDULES FOR DETERMINING
TOTAL MARKET DEMAND

Price/Quantity Schedule Adv. =$50,000		Advertising/Quantity Schedule Price=$10	
Price	Quantity	ADV.	% Change
$250	1	$ 50000	- *
$240	2	$ 70000	157.43%
$220	6	$ 90000	372.12%
$200	20	$110000	606.63%
$180	71	$130000	823.21%
$160	246	$150000	994.04%
$140	836	$170000	1104.47%
$120	2,766	$190000	1151.81%
$100	8,888	$210000	1086.12%
$ 80	25,597	$230000	996.98%
$ 60	82,259	$250000	943.94
$ 40	233,165		
20	621,492		
10	997,232		
1	1.773.357		

*percentage change computed on quantity as $50,000 being the base amount.

The interpolation method described in this paper is not intended to be a method that produces results identical to other mathematical models. The fact that the interpolation method give results which nearly duplicate values from more sophisticated mathematical methods gives the interpolation method considerable merit and appeal. The usefulness of the interpolation method in the final analysis does not depend on whether it can duplicate the results of other methods. Its validity and usefulness has to be judged in terms of whether it gives the simulation designer the desired results without undue complexity. The author has found the interpolation method described in this paper extremely valuable in the design of mathematical functions for his own simulations.

SUMMARY

An approach for approximating functional equations having minimum and maximum values and points of inflection has been presented. Using data from the multiplicative model presented by Gold and Pray, a demand model based on the interpolation process was used. The results were compared to data generated by the Gold and Pray model using identical price and advertising amounts. The differences in results did not vary acre than 12%. The interpolation approach does not involve complex mathematics but gives results comparable to methods that do. The major advantage of the interpolation method is that it gives the simulation designer control over desired results without the need for difficult-to--compute parameters required in models such as the Gold and Pray Demand model. Using the interpolation method presented in this paper, points of inflection along with minimum and maximum values are easy to create.

REFERENCES

/1/ Pray, Thomas F. and Steven Gold, "Inside the Black Box: An Analysis of Underlying Demand Functions in Contemporary Business Simulations", Developments in Business Simulations and Experiential Exercises, Vol. 9, 1982

/2/ Goosen, Kenneth R., "A Generalized Algorithm for Designing and Developing Business Simulations," Developments in Business Simulations and Experiential Exercises, Vol. 8, 1981.

/3/ Gold, Steven and Pray Thomas F., "Simulating Market And Firm Level Demand - A Robust Demand System", Developments in Business Simulations and Experiential Exercises, Vol. 10, 1983.

FIGURE 10
GRAPHICAL PRESENTATION OF SCHEDULE OF RANGE VALUES

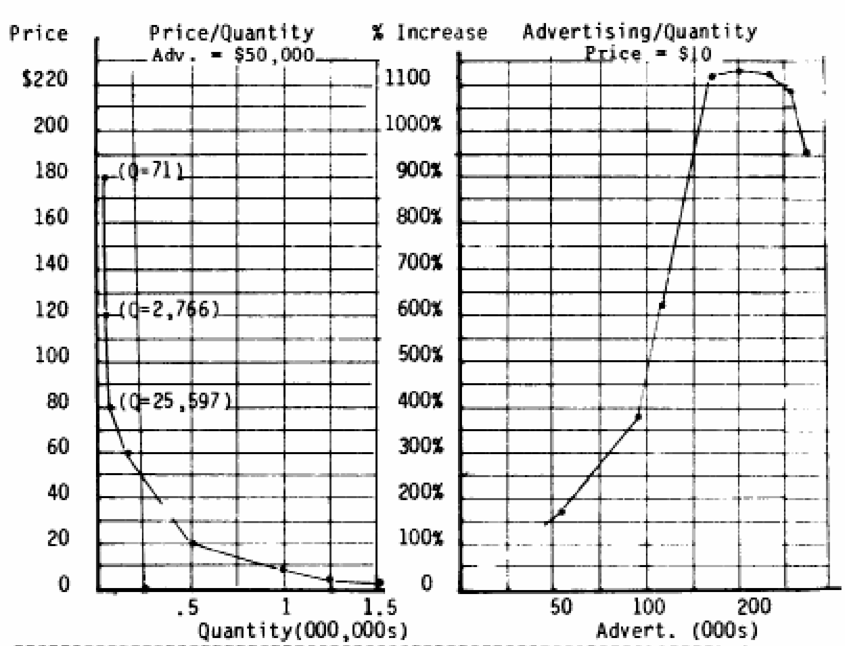


FIGURE 11

COMPARISON OF TOTAL MARKET DEMAND COMPUTED BY INTERPOLATION WITH DEMAND FROM THE GOLD/PRAY DEMAND MODEL

	Interpolation				
Price	Price Quantity* (PQ)	Adv. % Change** (PC)	Tot. Mkt. Demand ((P*(1+PC))	P/G Market Demand** (TQ)	% Difference
40	233165	606.63%	1,647,614	1,647,615	0.0000
38	271997	606.63%	1,922,012	1,822,651	0.0545
36	310830	606.63%	2,196,418	2,014,998	0.0900
34	349663	606.63%	2,470,823	2,226,200	0.1098
32	388495	606.63%	2,745,222	2,457,923	0.1160
30	427328	606.63%	3,019,627	2,711,966	0.1134
28	466161	606.63%	3,294,033	2,990,273	0.1015
26	504993	606.63%	3,568,432	3,294,957	0.0651
24	543826	606.63%	3,842,837	3,628,327	0.5812
22	582569	606.63%	4,116,607	3,992,937	0.0319
20	621492	606.64%	4,391,649	4,391,655	0.0000

*At advertising of $50,000

**At advertising of $110,000

APPENDIX A

INTERPOLATION MODEL FOR COMPUTING VALUES OF FUNCTIONAL EQUATIONS

Interpolaton is an operation for determining a number or value not listed in a table or array of numerical values by deriving the missing value on the basis of proportions computed from the numbers listed in a table. Mathematically, the procedure for computing an interpolated valued may be defined as:

$$IV = Y(i) + \left[\frac{X(i+1) - DV}{X(i+1) - X(i)} \right] (Y(i+1) - Y(i))$$

$i = 1, (N - 1)$

where:

IV = interpolated value (the dependent variable)
Y(i) = array of dependent values
X(i) = array of independent values
DV = Decision value (independent varible not in the X array
N = Number of items in each array

The equation is ideally suited for being solved by a computer program. A programmed algorithm to solve this equation in BASIC is presented in Appendix B.

APPENDIX B

BASIC COMPUTER PROGRAM FOR COMPUTING INTERPOLATED VALUES

The programming of the interpolation approach in this paper is fairly simple. Following is an example of the programming required to implement the interpolation method in an actual business simulation The data in this particular example is obtained from the illustration in the paper comparing the Gold/Pray demand model to the interpolation method. This particular program may be used to implement any graphical representation of a functional relationship. The only significant changes would be in statements 100, 110, 130, and 150. The interpolation method is generic, and consequently, only minor changes in the program below are required to make it applicable to other types of functional relationships.

```
10 CLS
29 DEFDBL A-H,O-Z
80 DIM P(20),Q(20),A(20),PT(20),X(20),Y(20)
90 FOR I = 1 TO 15:READ P(I):NEXT I
100 DATA 1,10,20,40,60,80,100,120,140,160,180,200,220,240,250
105 FOR I = 1 TO 15:READ Q(I):NEXT I
110 DATA 11773357,997232,621492,233165,82259,25597,8888,2766,836,246,71,20,6,2,1
120 FOR I = 1 TO 11:READ A(I):NEXT I
130 DATA 50000,70000,90000,110000,130000,150000,170000,190000,210000,230000,250000
140 FOR I = 1 TO 11:READ PT(I):NEXT I
150 DATA 0,158,372,607,823,994,1104,1152,1086,997,994
200 INPUT "PRICE DECISION";PD
210 INPUT "ADVERTISING DECISION";AD
500 REM PRICE
510 FOR I = 1 TO 15:X(I) = P(I):NEXT I
520 FOR I = 1 TO 15:Y(I) = Q(I):NEXT I
524 NI = 15
525 DV = PD
526 DV = PD
530 GOSUB 1000
550 PQ = IV
600 REM ADVERTISING
610 FOR I = 1 TO 11:X(I) = A(I):NEXT I
620 FOR I = 1 TO 11:Y(I) = PT(I):NEXT I
624 NI = 11
626 DV = AD
630 GOSUB 1000
650 PC = IV *.01
700 TQ = PQ * ( 1 + PCA)
800 CLS:PRINT "TOTAL DEMAND";TAB(18) "QUANTITY";TAB(29)"ADV. %"
810 PRINT USING"###,###,###.   ##,###,###.   #.####";TQ,PQ,PC
830 STOP
1000 REM INTERPOLATION SUBROUTINE
1010 FOR J = 1 TO NI
1020 IF DV >X(J) AND DV <= X (J+1) THEN TV = J
1030 IF TV > 0 GOTO 1390
1330 NEXT J
1390 GOSUB 4000
1400 IV = (((DV - LX)/(HX - LX))* (HY - LY)) + LY
1410 TV = 0:RETURN
4000 LX = X(TV + 1)
4010 HX = X(TV)
4020 LY = Y(TV + 1)
4030 HY = Y(TV)
4040 RETURN
```